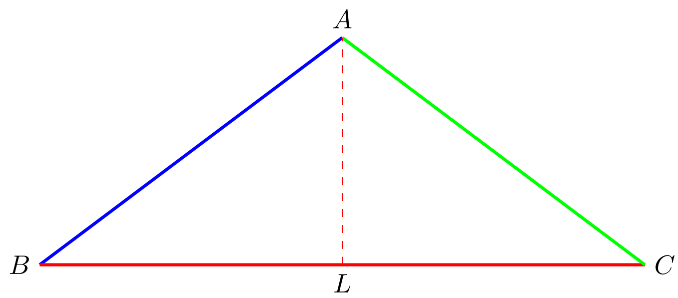
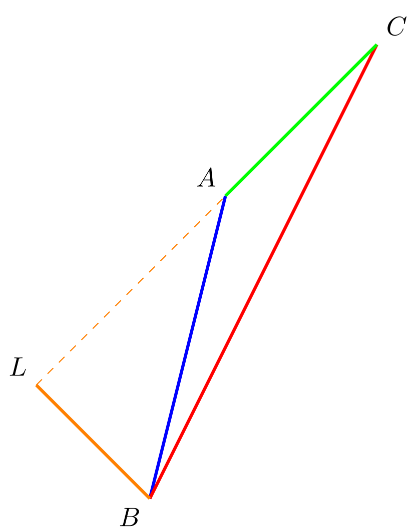
ALTITUDE of a triangle:

- a line segment from one vertex that is at right angles ($90°$) to the opposite side
- for example, in $\triangle ABC$ below, the altitude from vertex A is the line segment AL.

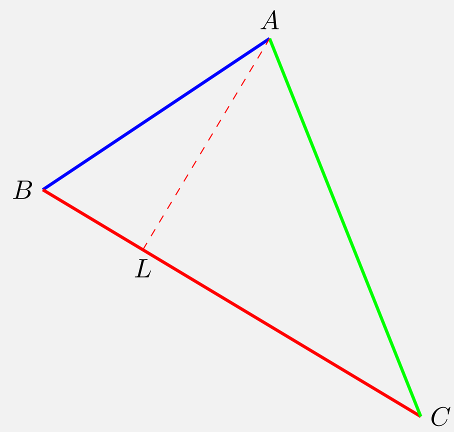


- sometimes the opposite side has to be extended before the altitude can meet it at right angles. For example, in $\triangle ABC$ below, the altitude from vertex B is the orange line segment BL. Notice how side CA was extended.



- the three altitudes of a triangle all meet at a point called the orthocenter, usually denoted by H.
- finding the equation of an altitude is easy. For example, for the altitude from vertex A:
  - obtain the **slope** of the opposite side (side BC in this case), then find its negative reciprocal;
  - use the coordinates of vertex A and the negative reciprocal slope in $y = mx + b$, where:
    - m is the negative reciprocal slope;
    - x and y come from the coordinates of vertex A.
  - easier DONE than SAID.

EXAMPLE 1: $\triangle ABC$ has vertices at $A(0,3)$, $B(-3,1)$, $C(2,-2)$. Find the equation of the altitude from vertex A.



- the slope of side BC is $\frac{-2-1}{2--3} = -\frac{3}{5}$, and its negative reciprocal is $\frac{5}{3}$. So we have $m = \frac{5}{3}$.
- the coordinates of vertex $A(0,3)$ are $x = 0$ and $y = 3$. Together with the value of m from above, the straight line equation $y = mx + b$ becomes:

$$y = mx + b$$
$$3 = \frac{5}{3}(0) + b$$
$$3 = b$$

- therefore, the equation of the altitude from vertex A is $\boxed{y = \frac{5}{3}x + 3}$. That's line AL above.

EXAMPLE 2: $\triangle ABC$ has vertices at $A(-2,-4)$, $B(1,5)$, $C(3,-4)$. Find the equation of the altitude from vertex B.

- the slope of side AC is ZERO, 0. So the altitude through vertex B has an undefined slope.
- any line with an undefined slope has an equation of the form $x = constant$. In our case, $x = 1$ (the x-coordinate of vertex B).
- therefore, the equation of the altitude from vertex B is $\boxed{x = 1}$.

EXAMPLE 3: $\triangle ABC$ has vertices at $A(0,6),\ B(0,-5),\ C(3,2)$. Find the equation of the altitude from vertex C.

- the slope of side AB is undefined. So the altitude through vertex C has a zero slope.
- any line with an zero slope has an equation of the form $y = constant$. In our case, $y = 2$ (the y-coordinate of vertex C).
- therefore, the equation of the altitude from vertex C is $\boxed{y = 2}$.

EXAMPLE 4: Find the orthocenter of $\triangle ABC$ with vertices at $A(0,0),\ B(1,4),\ C(3,6)$.

- we need to find the equations of any two altitudes, and then solve the two equations simultaneously.
- let's start with the equation of the altitude from vertex A:
  - the slope of side BC is $\frac{6-4}{3-1} = \frac{2}{2} = 1$, and its negative reciprocal is -1. Take $m = -1$.
  - the coordinates of vertex $A(0,0)$ are $x = 0$ and $y = 0$. Together with the value of m, the straight line equation $y = mx + b$ becomes

$$0 = -1(0) + b \implies b = 0$$

  - so we get $\boxed{y = -x}$ as the equation of the altitude from A.
- now let's find the equation of the altitude from vertex B:
  - the slope of side AC is $\frac{6-0}{3-0} = 2$, and its negative reciprocal is $-\frac{1}{2}$. Take $m = -\frac{1}{2}$.
  - the coordinates of vertex $B(1,4)$ are $x = 1$ and $y = 4$. Together with $m = -\frac{1}{2}$, the straight line equation $y = mx + b$ becomes:

$$4 = -\frac{1}{2}(1) + b \implies 4 + \frac{1}{2} = b \implies \frac{9}{2} = b$$

  - so we get $\boxed{y = -\frac{1}{2}x + \frac{9}{2}}$ as the equation of the altitude from vertex B.
- finally we solve the two altitude equations (namely, $y = -x$ and $y = -\frac{1}{2}x + \frac{9}{2}$) simultaneously:

$$-x = -\frac{1}{2}x + \frac{9}{2} \implies -2x = -x + 9 \implies x = -9 \qquad \therefore y = -x = -(-9) = 9$$

- conclusion: the orthocenter is located at $(-9, 9)$.

EXAMPLE 5: Find the orthocenter of $\triangle ABC$ with vertices at $A(2,5),\ B(-2,0),\ C(4,1)$.

- as before, we find the equations of any two altitudes, and then solve the resulting system of linear equations.
- let's start with the equation of the altitude from vertex A:
  - the slope of side BC is $\frac{1-0}{4--2} = \frac{1}{6}$, and its **negative reciprocal** is -6. Take $m = -6$, to be used in the straight line equation $y = mx + b$.
  - the coordinates of vertex $A(2,5)$ are $x = 2$ and $y = 5$. In the straight line equation $y = mx + b$, we have:

$$\begin{aligned} y &= mx + b \\ 5 &= -6(2) + b \\ 5 &= -12 + b \\ 17 &= b \end{aligned}$$

  - the altitude from vertex A has equation $y = -6x + 17$
- next, let's find the equation of the altitude from vertex B:
  - the slope of side AC is $\frac{5-1}{2-4} = -2$, and its **negative reciprocal** is $\frac{1}{2}$. Take $m = \frac{1}{2}$, to be used in the straight line equation $y = mx + b$.
  - the coodinates of vertex $B(-2,0)$ are $x = -2$ and $y = 0$. Together with $m = \frac{1}{2}$, the straight line equation $y = mx + b$ becomes:

$$\begin{aligned} y &= mx + b \\ 0 &= \frac{1}{2}(-2) + b \\ 0 &= -1 + b \\ 1 &= b \end{aligned}$$

  - the altitude from vertex B has equation $y = \frac{1}{2}x + 1$.
- finally, we solve the linear system $y = -6x + 17,\ y = \frac{1}{2}x + 1$ to determine the orthocenter:

$$\begin{aligned} -6x + 17 &= \frac{1}{2}x + 1 \\ -12x + 34 &= x + 2 \\ -13x &= -32 \\ \therefore x &= \frac{32}{13} \end{aligned}$$

$$\cdots\cdots\cdots$$

$$\begin{aligned} y &= \frac{1}{2}x + 2 \\ &= \frac{1}{2}\left(\frac{32}{13}\right) + 1 \\ &= \frac{16}{13} + 1 \\ &= \frac{29}{13} \end{aligned}$$

- the orthocenter is at $\left(\frac{32}{13}, \frac{29}{13}\right)$.