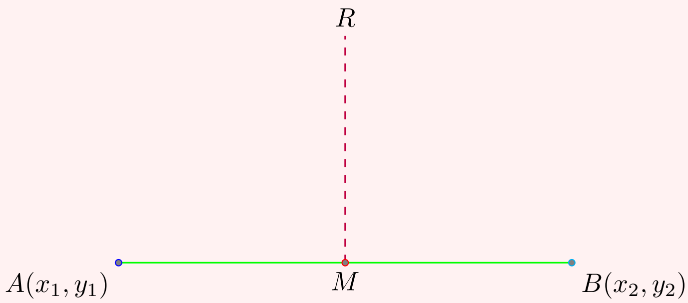
LESSON 9: Equation of a right bisector

Beautiful analytic geometry continues ···

- a right bisector of a line segment is a line through the midpoint of the line segment and which makes a $90°$ (right) angle with it. The term right bisector can also be rendered as perpendicular bisector.



- for example, in the diagram above, MR is the right bisector of the line segment AB.
- the three right bisectors of the sides of a triangle *concur* at a point called the circumcenter – usually denoted by O.
- finding the equation of a right bisector is easy:
  - obtain the midpoint of the line segment, using the midpoint formula $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$;
  - obtain the slope of the line segment (using $\frac{y_2-y_1}{x_2-x_1}$);
  - obtain the slope of the right bisector, which is just the negative reciprocal of the slope obtained previously;
  - obtain the right bisector equation by using the slope of the right bisector and the midpoint in the straight line equation $y = mx + b$.
- Easier *done* than *said*.

Then, don't forget to get ···

Those. two. cups. will. do. for. this. discourse.

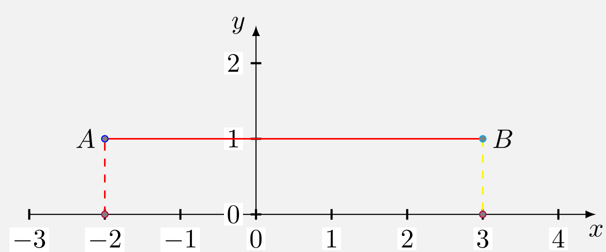
EXAMPLE 1: Find the equation of the right bisector of the line segment AB, given $A(0,1)$ and $B(2,7)$.

- the midpoint of AB is $\left(\frac{0+2}{2}, \frac{1+7}{2}\right) = (1,4)$. Take note of this; it will be used shortly.
- the slope of AB is $\frac{7-1}{2-0} = \frac{6}{2} = 3$. Hence, the slope of the right bisector is $-\frac{1}{3}$ (remember: negative reciprocal).
- next, we use $y = mx + b$ at the midpoint $(1,4)$ and $m = -\frac{1}{3}$:

$$\begin{aligned} y &= mx + b \\ 4 &= -\frac{1}{3}(1) + b \\ 12 &= -1 + 3b \\ 13 &= 3b \\ \frac{13}{3} &= b \end{aligned}$$

- therefore, the equation of the right bisector is $\boxed{y = -\frac{1}{3}x + \frac{13}{3}}$.

EXAMPLE 2: Find the equation of the right bisector of the line segment AB in the diagram below:



From the diagram, A is the point $(-2,1)$, while B is the point $(3,1)$.

- the midpoint of AB is $\left(\frac{-2+3}{2}, \frac{1+1}{2}\right) = \left(\frac{1}{2}, 1\right)$. Note this down; we'll use it shortly.
- the slope of AB is zero; it is a horizontal line. Hence the slope of the right bisector, being the *negative reciprocal of the slope of AB*, is undefined. In other words, the right bisector in this case is a vertical line.
- since the right bisector goes through $\left(\frac{1}{2}, 1\right)$ and is a vertical line, its equation is $\boxed{x = \frac{1}{2}}$.

EXAMPLE 3: Points A and B are located at $A(-2, 3)$ and $B(4, 11)$. Find the equation of the right bisector of the line segment AB.

- the midpoint of AB is $\left(\frac{-2+4}{2}, \frac{3+11}{2}\right) = (1, 7)$.
- the slope of AB is $\frac{11-3}{4--2} = \frac{8}{6} = \frac{4}{3}$. Hence, the slope of the right bisector is $-\frac{3}{4}$.
- next we use $y = mx + b$ at the midpoint $(1, 7)$ and with $m = -\frac{3}{4}$:

$$\begin{aligned} y &= mx + b \\ 7 &= -\frac{3}{4}(1) + b \\ 28 &= -3 + 4b \\ 31 &= 4b \\ \frac{31}{4} &= b \end{aligned}$$

- therefore, the equation of the right bisector is $\boxed{y = -\frac{3}{4}x + \frac{31}{4}}$.

EXAMPLE 4: Find the equation of the right bisector of the line segment joining the points $A(0, 3)$ and $B(0, -5)$.

Observe that AB is a vertical line, so its right bisector will be a horizontal line.

As in EXAMPLE 2, the calculations in this case is simplified; we only need to find the midpoint of the line segment and then utilize the fact that the right bisector is a horizontal line. Here, the midpoint is $\left(\frac{0+0}{2}, \frac{3+-5}{2}\right) = (0, -1)$.

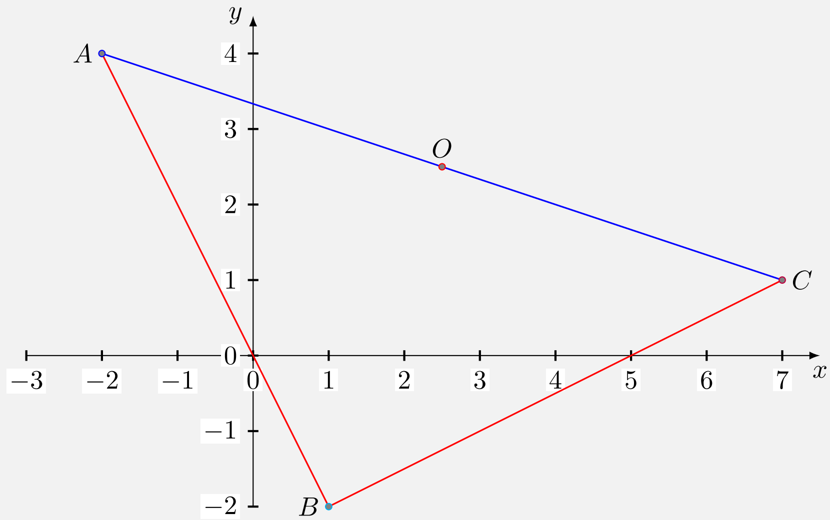
therefore, the equation of the right bisector is $\boxed{y = -1}$.

Special right bisectors

EXAMPLE 2 and EXAMPLE 4 demonstrate two special cases of right bisectors.

- if the given line segment is horizontal, then its right bisector will be vertical, and its equation will be of the form $\boxed{x = \frac{x_1 + x_2}{2}}$, that is, the x-coordinate of the midpoint.
- if the given line segment is vertical, then its right bisector will be horizontal, and its equation will be of the form $\boxed{y = \frac{y_1 + y_2}{2}}$, that is, the y-coordinate of the midpoint.
- a horizontal line has equal y-coordinates; a vertical line has equal x-coordinates.

EXAMPLE 5: Find the circumcenter of the triangle ABC formed by the points $A(-2,4)$, $B(1,-2)$, $C(7,1)$.



We will need to find the equations of any two right bisectors, and then solve the equations simulataneously.

- Let's start with the right bisector of side AB. The midpoint of AB is at $\left(\frac{-2+1}{2}, \frac{4+-2}{2}\right) = \left(-\frac{1}{2}, 1\right)$. The slope of AB is $\frac{-2-4}{1--2} = \frac{-6}{3} = -2$. Hence, the slope of the right bisector of AB is $\frac{1}{2}$ (remember: *negative reciprocal*).
  - using $m = \frac{1}{2}$ and the midpoint $\left(-\frac{1}{2}, 1\right)$ in the linear equation $y = mx + b$:

$$1 = \frac{1}{2}\left(-\frac{1}{2}\right) + b \implies 1 = -\frac{1}{4} + b \implies b = 1 + \frac{1}{4} = \frac{5}{4}$$

  - the equation of the right bisector of side AB is $\boxed{y = \frac{1}{2}x + \frac{5}{4}}$

- Next, the midpoint of BC is $\left(4, -\frac{1}{2}\right)$, and its slope is $\frac{1}{2}$, meaning that the slope of the right bisector of side BC is -2. Thus, the equation of the right bisector of side BC is $\boxed{y = -2x + \frac{15}{2}}$.

- Set the two right bisector equations equal to each other:

$$\begin{aligned} \frac{1}{2}x + \frac{5}{4} &= -2x + \frac{15}{2} \\ 2x + 5 &= -8x + 30 \\ \therefore x &= \frac{5}{2} \\ \therefore y &= \frac{5}{2} \end{aligned}$$

Thus, the circumcenter is at $\left(\frac{5}{2}, \frac{5}{2}\right)$ – marked as O in the diagram above. Notice that this is precisely the midpoint of side AC. Always the case for right triangles: *the circumcenter of a right triangle is at the midpoint of the hypotenuse.*