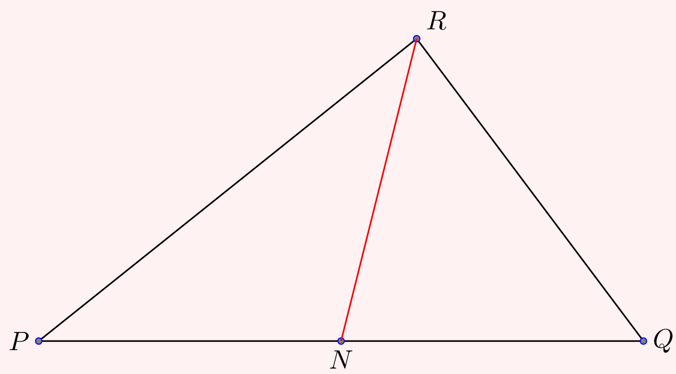
LESSON 8: Equation (and length) of a median

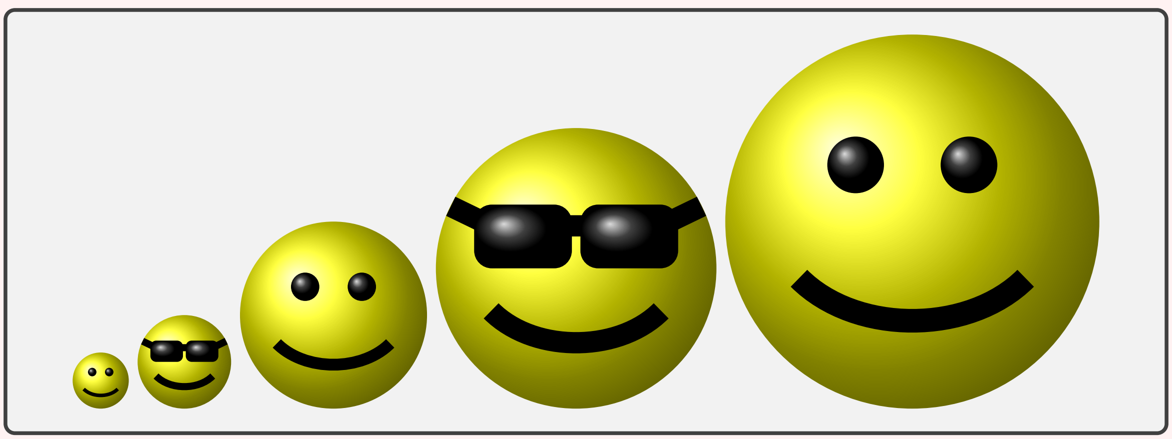
Beautiful analytic geometry continues ···

- in a triangle, a median is a line segment from one vertex to the midpoint of the opposite side. For example, in $\triangle PQR$ below, the median from vertex R is the red line segment RN:



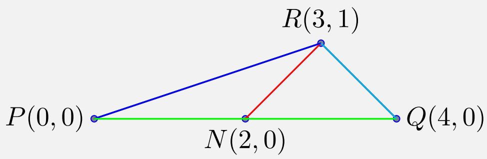
- a triangle has three medians, and they all intersect at a point called the centroid of the triangle (usually denoted by G – think center of "**g**ravity").
- since a median is a line segment, we can find its length, its equation, its midpoint.
- if a triangle has coordinates at (x_1, y_1), $(x_2, y_2), (x_3, y_3)$, then its centroid is located at $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.
- Easy-peasy.

As always: $\cdots \ddots \vdots$



As always.

EXAMPLE 1: $\triangle PQR$ has vertices at $P(0,0)$, $Q(4,0)$, and $R(3,1)$. Find the length of the median from vertex R.

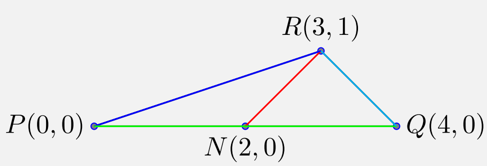


Since we need the median from vertex R, we first find the midpoint of side PQ, which is $\left(\frac{0+4}{2}, \frac{0+0}{2}\right) = (2,0)$. This is the point N marked above. Thus, the length of the median RN is, by the distance formula:

$$\begin{aligned} RN &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3-2)^2 + (1-0)^2} \\ &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \end{aligned}$$

The length of median RN is $\sqrt{2}$ units.

EXAMPLE 2: $\triangle PQR$ has vertices at $P(0,0)$, $Q(4,0)$, and $R(3,1)$. Find the equation of the median from vertex R.



This is the same example as before, just that we need the equation of median RN this time around. The **slope** of RN is

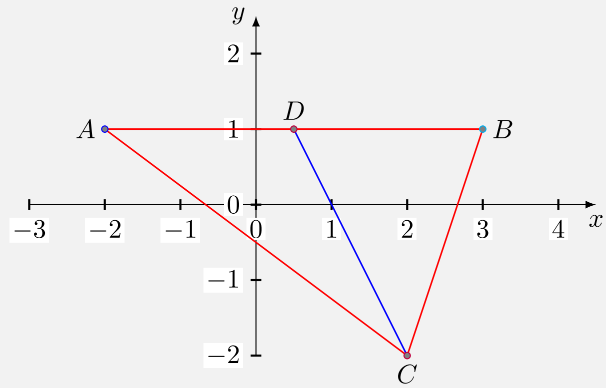
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{3-2} = 1.$$

The equation of a line in terms of the **slope** m and the y-intercept b is $y = mx + b$. In this case, $m = 1$, so we have $y = x + b$. It remains to find b. We can use either the coordinates of R or the coordinates of N. Let's use $N(2,0)$:

$$y = x + b \implies 0 = 2 + b \implies 0 - 2 = b \implies -2 = b.$$

Thus, $y = x - 2$ is the equation of median RN.

EXAMPLE 3: Find the midpoint of the median from vertex C in the diagram below:

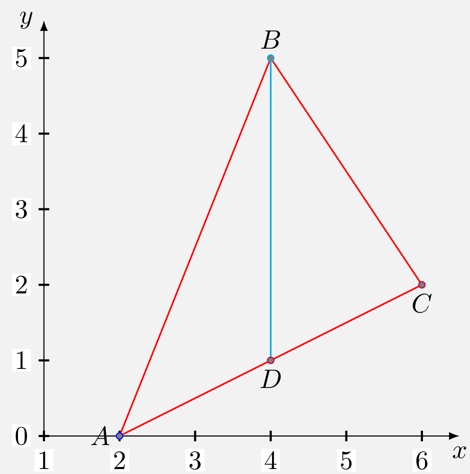


The median from vertex C is the line segment CD shown above. Observe that A is the point $(-2, 1)$, while B is the point $(3, 1)$. Thus, D, being the midpoint of A and B, is the point $\left(\frac{-2+3}{2}, \frac{1+1}{2}\right) = \left(\frac{1}{2}, 1\right)$.

Since C is the point $(2, -2)$, the midpoint of CD is then:

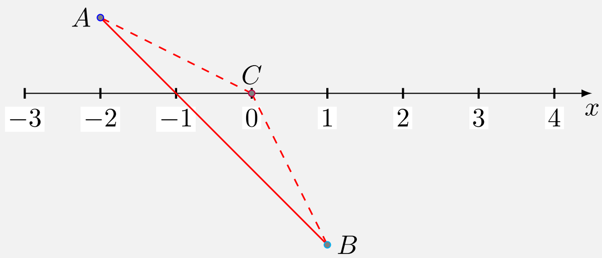
$$\left(\frac{2+\frac{1}{2}}{2}, \frac{-2+1}{2}\right) = \left(\frac{5/2}{2}, \frac{-1}{2}\right) = \left(\frac{5}{4}, -\frac{1}{2}\right).$$

EXAMPLE 4: Find the equation of the median from vertex B in the diagram below:



From the above diagram, the vertices of $\triangle ABC$ are $A(2, 0)$, $B(4, 5)$, $C(6, 2)$. Since we need the equation of the median from vertex B, we need to first find the midpoint of AC, which is $\left(\frac{2+6}{2}, \frac{0+2}{2}\right) = (4, 1)$. The median BD has endpoints $B(4, 5)$ and $D(4, 1)$. It is a vertical line. Thus, its equation is $x = 4$.

EXAMPLE 5: Find the **centroid** of $\triangle ABC$ formed by the points $A(-2,1),\ B(1,-2),\ C(0,0)$.



Ideally, one should use linear systems here: find the equations of any two medians and then solve the two equations simultaneously to obtain their point of intersection, which would then be the **centroid**.

Instead, we use the short-cut $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = \left(\frac{-2+1+0}{3}, \frac{1+-2+0}{3}\right) = \left(-\frac{1}{3}, -\frac{1}{3}\right)$, which is basically the *average* of the x-coordinates and the *average* of the y-coordinates, taken separately.

The centroid is $\left(-\frac{1}{3}, -\frac{1}{3}\right)$.

Use **linear systems** to *verify*, and then this: