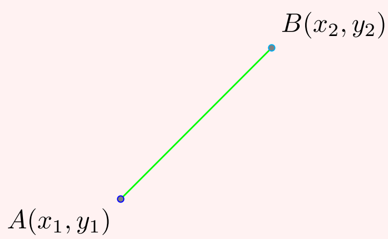
LESSON 6: Length of a line segment

WELCOME to analytic geometry!

✍ a line segment has two distinct endpoints; for example, in the diagram below, AB is a line segment whose endpoints are A and B:

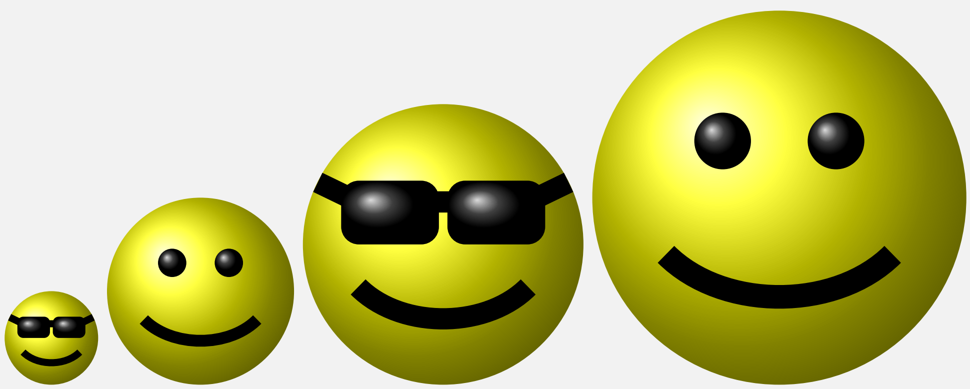


✍ the length of line segment AB with coordinates given above is:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

✍ Easy-peasy.

$\ldots \ddots \vdots$



As always.

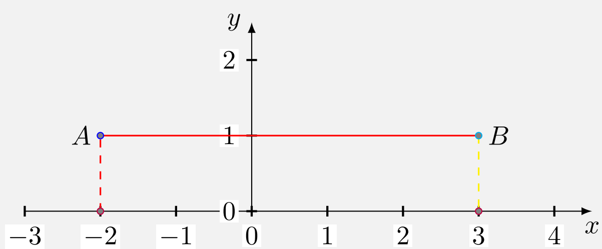
EXAMPLE 1: Find the length of the line segment joining the points $A(0,1)$ and $B(2,7)$.

We use equation (1) with $A(x_1, y_1) = A(0,1)$ and $B(x_2, y_2) = B(2,7)$. Basically, $x_1 = 0,\ y_1 = 1$ and $x_2 = 2,\ y_2 = 7$. So:

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2-0)^2 + (7-1)^2} \\ &= \sqrt{2^2 + 6^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= \sqrt{4}\sqrt{10} \\ &= 2\sqrt{10} \end{aligned}$$

The length of the line segment is $2\sqrt{10}$ units.

EXAMPLE 2: Find the length of the line segment AB in the diagram below:



From the diagram, A is the point $(-2,1)$, while B is the point $(3,1)$. Put $(x_1, y_1) = (-2,1)$ and $(x_2, y_2) = (3,1)$ in equation (1):

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - -2)^2 + (1-1)^2} \\ &= \sqrt{5^2 + 0^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

The length of the line segment AB is 5 units.

EXAMPLE 3: A circle has one diameter with endpoints at $A(-2, 3)$ and $B(4, 11)$. Find the length of the radius of the circle.

The length of the radius is **half** the length of the diameter of a circle. Let's first find the diameter's length, which is the same as the length of AB. Put $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, 11)$ in equation (1):

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - -2)^2 + (11 - 3)^2} \\ &= \sqrt{6^2 + 8^2} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

The diameter is 10 units, and so the radius is 5 units.

EXAMPLE 4: A is the point $(a, -5)$, while B is the point $(0, 7)$. Find the value(s) of a for which the length of the line segment AB is 13 units.

Put $(x_1, y_1) = (a, -5)$ and $(x_2, y_2) = (0, 7)$ in equation (1):

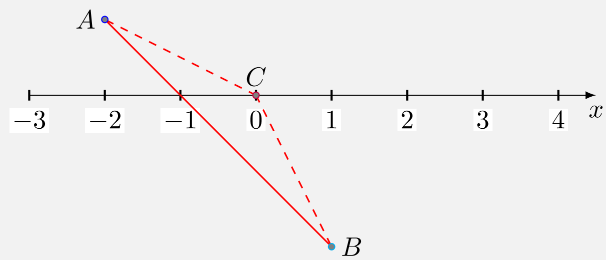
$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ 13 &= \sqrt{(0 - a)^2 + (7 - -5)^2} \\ 13 &= \sqrt{(-a)^2 + 12^2} \\ 13 &= \sqrt{a^2 + 144} \end{aligned}$$

Now, square both sides of the equation to eliminate the square root:

$$\begin{aligned} 13^2 &= a^2 + 144 \\ 169 &= a^2 + 144 \\ 169 - 144 &= a^2 \\ 25 &= a^2 \\ \therefore \pm 5 &= a \end{aligned}$$

Thus, $a = 5$ or $a = -5$.

EXAMPLE 5: Use the distance formula to classify the triangle ABC formed by the points $A(-2,1)$, $B(1,-2)$, $C(0,0)$.



We apply the distance formula repeatedly to find the lengths of AB, BC, CA:

$$\begin{aligned}
AB &= \sqrt{(1--2)^2+(-2-1)^2} \\
&= \sqrt{3^2+(-3)^2} \\
&= \sqrt{9+9} \\
&= \sqrt{18} \\
&= \sqrt{9\times 2} \\
&= 3\sqrt{2} \\
&\dots\vdots \\
BC &= \sqrt{(0-1)^2+(0--2)^2} \\
&= \sqrt{(-1)^2+2^2} \\
&= \sqrt{1+4} \\
&= \sqrt{5} \\
&\dots\vdots \\
CA &= \sqrt{(-2-0)^2+(1-0)^2} \\
&= \sqrt{(-2)^2+1^2} \\
&= \sqrt{4+1} \\
&= \sqrt{5}
\end{aligned}$$

Since $BC = CA$, the triangle is isosceles.

Don't forget this: