LESSON 12: Equation of a circle

Beautiful analytic geometry ends here $\cdots$ for now $\cdots$

- ✍ the equation of a circle with center at the origin $(0, 0)$ and radius r is given by:

$$\boxed{x^2 + y^2 = r^2}$$

- ✍ if the center is moved to the point (h, k), then the circle equation becomes:

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}.$$

- ✍ notice how the first equation is a special case of the second equation (by putting $h = 0,\ k = 0$).
- ✍ Easy stuff.

EXAMPLE 1: Find the equation of a circle with center at the origin and radius 4 units.

Use $x^2 + y^2 = r^2$ and put $r = 4$ to obtain $x^2 + y^2 = 16$.

Is that all? NO! It remains this:

That's it!

EXAMPLE 2: Find the radius of a circle whose equation is $x^2 + y^2 = 121$.

Compare with the standard equation $x^2 + y^2 = r^2$ and obtain $r^2 = 121$. Since the radius is a length, we take the positive square root: $r = \sqrt{121} = 11$.

EXAMPLE 3: One diameter of a circle has end points at $A(-3, 4)$ and $B(3, -4)$. Find the equation of the circle.

We first need to find the center, and the radius, of the circle.

- center: mid point of a diameter: $\left(\frac{-3+3}{2}, \frac{4+-4}{2}\right) = (0, 0)$. The center is at $(0, 0)$, the origin.
- radius: since we already know the coordinates of the center and the end points of a diameter, we can find the radius using the distance formula. Alternatively, we could find the length of the diameter and take its half as the radius length. Let's do the former:

$$\begin{aligned} \text{radius} \quad r &= \sqrt{(-3-0)^2 + (4-0)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

  The circle equation $x^2 + y^2 = r^2$ uses r^2, we might as well not bother with the square root above.
- circle equation: combining the center $(0, 0)$ and radius $r = 5$ we obtain the equation $x^2 + y^2 = 25$.

EXAMPLE 4: Given the circle $x^2 + y^2 = 50$ and the points $X(-1, 7)$ and $Y(-7, 1)$. Verify that XY is a chord of the circle.

A chord of a circle is a line segment which connects any two points on the circle. Thus, we have to show that both X and Y are points on the circle. This is easy:

$$\begin{aligned} \text{at } X(-1,7) &\implies (-1)^2 + 7^2 = 1 + 49 = 50 \\ &\qquad \therefore \text{X lies on the circle} \\ \text{at } Y(-7,1) &\implies (-7)^2 + 1^2 = 49 + 1 = 50 \\ &\qquad \therefore \text{Y lies on the circle} \end{aligned}$$

We conclude that XY is indeed a chord of this circle.

EXAMPLE 5: Find the equation of a circle with center $(-2,3)$ and passing through $(2,-1)$.

Note that the center is no longer at the origin, so we use a translated form of the circle equation, namely: $(x-h)^2+(y-k)^2=r^2$, where $(h,k)=(-2,3)$ is the new center.

We also need the radius, which is the distance from the center to any point on the circumference; in this case, the radius is the distance from $(-2,3)$ to $(2,-1)$:

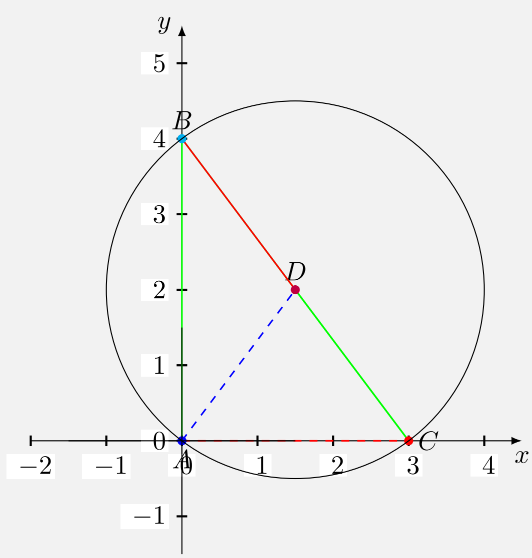
$$r=\sqrt{(2--2)^2+(-1-3)^2}=\sqrt{32}.$$

Therefore, the equation of this circle is $(x+2)^2+(y-3)^2=32$.

EXAMPLE 6: Find the equation of a circle which passes through the points $A(0,0)$, $B(0,4)$, $C(3,0)$.

As shown below, the three given points form a triangle – a right triangle for that matter. This means that the center of the circle – which is the circumcenter of the triangle – is easy to find in this case. From what we know about right triangles, the circumcenter of a right triangle is at the mid-point of the hypotenuse. Thus:

$$\begin{aligned}\text{center (midpoint of hypotenuse BC, labelled D in the diagram)} &= \left(\frac{0+3}{2},\frac{4+0}{2}\right)\\ &= (1.5,2)\\ \text{radius (distance DA or distance DB or distance DC)} &= 2.5\end{aligned}$$



Therefore, $(x-1.5)^2+(y-2)^2=6.25$ is the desired equation.

EXAMPLE 7: Verify that the point $(-3, 9)$ is inside the circle $x^2 + y^2 = 100$.

If a given point is to be inside a circle, then the distance from the point to the center of the circle must be less than the radius.

The distance from $(-3, 9)$ to the center $(0, 0)$ of the circle is:

$$d = \sqrt{(-3-0)^2 + (9-0)^2} = \sqrt{90}.$$

Since the radius of the circle is $r = \sqrt{100}$, we see that the above distance is less than the radius. Therefore the point $(-3, 9)$ lies inside the circle.