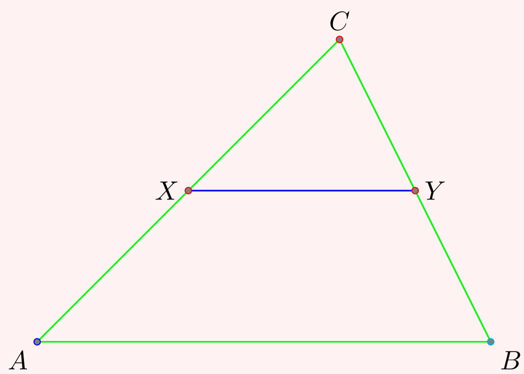
LESSON 10: The midpoint theorem

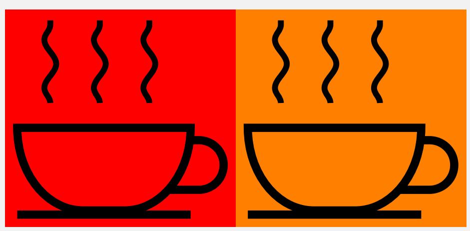
Beautiful analytic geometry continues ···

✍ midpoint theorem: a line which connects the midpoints of two sides of a triangle is parallel to the third side, and its length is equal to half of the length of the third side.



✍ for example, in the diagram above, X is the midpoint of side AC and Y is the midpoint of side BC. According to the midpoint theorem, the line segment XY is parallel to side AB, and in terms of length we have $XY = \frac{1}{2}AB$.

Easy stuff, with or without one of these cups ···



EXAMPLE 1: $\triangle ABC$ has vertices at $A(0,0)$, $B(6,0)$, $C(4,4)$. Let X and Y be the midpoints of sides AC and BC, respectively. Verify that XY is parallel to AB.

- the midpoint of AC is $\left(\frac{0+4}{2}, \frac{0+4}{2}\right) = (2,2)$. Thus, X is the point $(2,2)$.
- the midpoint of BC is $\left(\frac{6+4}{2}, \frac{0+4}{2}\right) = (5,2)$. Thus, Y is the point $(5,2)$.
- the slope of XY is $\frac{2-2}{5-2} = 0$; similarly, the slope of AB is $\frac{0-0}{6-0} = \frac{0}{6} = 0$.
- since XY and AB have the same slope, they are parallel.

EXAMPLE 2: $\triangle ABC$ has vertices at $A(0,0)$, $B(6,0)$, $C(4,4)$. Let X and Y be the midpoints of sides AC and BC, respectively. Verify that $XY = \frac{1}{2}AB$.

- the midpoint of AC is $\left(\frac{0+4}{2}, \frac{0+4}{2}\right) = (2,2)$. Thus, X is the point $(2,2)$.
- the midpoint of BC is $\left(\frac{6+4}{2}, \frac{0+4}{2}\right) = (5,2)$. Thus, Y is the point $(5,2)$.
- the length of XY is, by the distance formula:

$$XY = \sqrt{(5-2)^2 + (2-2)^2} = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$$

- the length of AB is, by the distance formula:

$$AB = \sqrt{(6-0)^2 + (0-0)^2} = \sqrt{6^2 + 0^2} = \sqrt{36} = 6$$

- since $XY = 3$ and $AB = 6$, we arrive at the desired conclusion: $XY = \frac{1}{2}AB$.

EXAMPLE 3: $\triangle ABC$ has vertices at $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$. Let X and Y be the midpoints of AB and AC, respectively. PROVE that XY is parallel to BC.

- the midpoint of AB is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Mark this as point X.
- the midpoint of AC is $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$. Mark this as point Y.
- the slope of XY is

$$\frac{\frac{y_1+y_3}{2} - \frac{y_1+y_2}{2}}{\frac{x_1+x_3}{2} - \frac{x_1+x_2}{2}} = \frac{\frac{y_3-y_2}{2}}{\frac{x_3-x_2}{2}} = \frac{y_3 - y_2}{x_3 - x_2}$$

- the slope of side BC is $\frac{y_3-y_2}{x_3-x_2}$.
- since XY and BC have equal slopes, they are *parallel*.

EXAMPLE 4: $\triangle ABC$ has vertices at $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$. Let X and Y be the midpoints of AB and AC, respectively. PROVE that $XY = \frac{1}{2}BC$.

- the midpoint of AB is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Mark this as point X.
- the midpoint of AC is $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$. Mark this as point Y.
- the length of XY is:

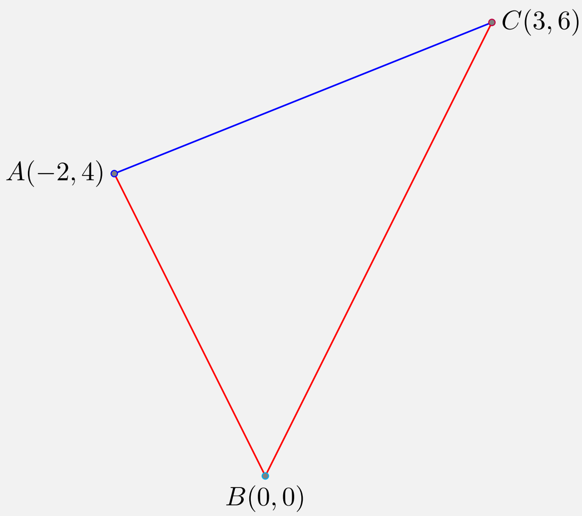
$$
\begin{aligned}
XY &= \sqrt{\left(\frac{x_1+x_3}{2} - \frac{x_1+x_2}{2}\right)^2 + \left(\frac{y_1+y_3}{2} - \frac{y_1+y_2}{2}\right)^2} \\
&= \sqrt{\left(\frac{x_3-x_2}{2}\right)^2 + \left(\frac{y_3-y_2}{2}\right)^2} \\
&= \sqrt{\frac{(x_3-x_2)^2 + (y_3-y_2)^2}{4}} \\
&= \frac{\sqrt{(x_3-x_2)^2 + (y_3-y_2)^2}}{2}
\end{aligned}
$$

- the length of side BC is $\sqrt{(x_3-x_2)^2 + (y_3-y_2)^2}$.
- we can now conclude that $XY = \frac{1}{2}BC$.

The medial triangle

The medial triangle of a given triangle is the triangle obtained by joining the midpoints of the three sides of the parent triangle.

EXAMPLE 5: Find the vertices of the medial triangle of the triangle ABC formed by the points $A(-2,4),\ B(0,0),\ C(3,6)$.



We just have to find the midpoints of the three sides.

- The midpoint of AB is at $\left(\frac{-2+0}{2}, \frac{4+0}{2}\right) = (-1,2)$.
- The midpoint of BC is $\left(\frac{0+3}{2}, \frac{0+6}{2}\right) = \left(\frac{3}{2}, 3\right)$.
- The midpoint of CA is $\left(\frac{-2+3}{2}, \frac{4+6}{2}\right) = \left(\frac{1}{2}, 5\right)$.
- Thus, the medial triangle has vertices at: $(-1,2)$, $\left(\frac{3}{2}, 3\right)$, $\left(\frac{1}{2}, 5\right)$.